WHEN IS A RIGHT ALTERNATIVE LOOP RING THAT IS ALSO RIGHT BOL ACTUALLY MOUFANG?

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Abstract. In this paper, we consider some connections between loops whose loop rings, in characteristic 2, satisfy the Moufang identities and loops whose loop rings, in characteristic 2, satisfy the right Bol identities.

1. Introduction

In [Goo83], the second author asked "When is a loop ring alternative?" This question was completely settled in characteristic different from 2 by the first two authors, who showed that when the coefficient ring R has characteristic different from 2, the loop ring RL is alternative if and only if L is a Moufang loop with commutator/associator subloop of order 2 and the property that two elements commute if and only if either one of them or their product is central [CG86]. They called such loops *RA loops*.

What about the remaining case that $\operatorname{char} R = 2$? This situation was studied in [?] where the term *RA2 loop* was coined to describe a loop whose loop rings in characteristic 2 are alternative.

Study of the next question, "Are there loops whose loop rings satisfy just the right alternative law?" started in [CG88] where the first two authors showed that if $\operatorname{char} R \neq 2$ and L is finite, or if $\operatorname{char} R = 2$ and L is Moufang, then RL satisfies the right alternative law if and only if it satisfies the left alternative law and, hence, is an alternative ring. In 1998, K. Kunen dropped the finiteness condition for the first result, and addressed the obvious question raised by the second: "What if $\operatorname{char} R = 2$ and L is not Moufang?" He produced an example of a loop L that was not even right Bol, for which RL satisfies the right but not the left alternative law in characteristic 2.

Since alternative rings satisfy the Moufang identities, if a loop ring RL is alternative, then L must be Moufang. As Kunen's example shows, however, in characteristic 2, if RL is right (but not left) alternative then not only is it not necessary that RL satisfy the right Bol identity, but L need not even be Bol. This prompted the second author and D. A. Robinson to ask, and partially answer, "When is a right alternative loop ring in characteristic 2 satisfy the right Bol identity?" In [GR95], they proved the following theorem:

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Theorem 1.1. *If L is a loop and R is a commutative ring with unity and with* $\operatorname{char} R = 2$, *then the loop ring RL satisfies the right Bol identity if and only if L is a right Bol loop and, for all $x, y, z, w \in L$, at least one of the following three conditions holds:*

$$\begin{aligned} &D(x,y,z,w) : [(xy)z]w = x[(yz)w] \text{ and } [(xw)z]y = x[(wz)y] \\ &E(x,y,z,w) : [(xy)z]w = x[(wz)y] \text{ and } [(xw)z]y = x[(yz)w] \\ &F(x,y,z,w) : [(xy)z]w = [(xw)z]y \text{ and } x[(yz)w] = x[(wz)y]. \end{aligned} \tag{1.1}$$

They called Bol loops which satisfy these conditions *strongly right alternative ring loops*, *SRAR loops* for short.[1]

In the remainder of this paper, for brevity, when there is no likelihood of confusion, we will simply say that an SRAR loop satisfies D or E or F.

Note that conditions D, E and F involve only four different expressions: $[(xy)z]w$, $x[(yz)w]$, $[(xw)z]y$ and $x[(wz)y]$. Denoting these four expressions by S, T, U and V, the conditions D, E and F are of the form

$$\begin{gathered} S = T \text{ and } U = V \\ S = U \text{ and } T = V \\ S = V \text{ and } T = U. \end{gathered}$$

Clearly, if any two of the three conditions hold, then

$$S = T = U = V,$$

and so all three conditions hold.

In other words, we have the following lemma:

Lemma 1.2. *If x, y, z, w are any four elements of an SRAR loop, then either*

i) $D(x,y,z,w)$, $E(x,y,z,w)$ and $F(x,y,z,w)$ all hold, or else
ii) only $D(x,y,z,w)$ holds, or
iii) only $E(x,y,z,w)$ holds, or
iv) only $F(x,y,z,w)$ holds.

Note also that, if we set $w = 1$ in the conditions D, E and F, we get the following conditions:

$$\begin{aligned} &D'(x,y,z) : (xy)z = x(yz) \text{ and } (xz)y = x(zy) \\ &E'(x,y,z) : (xy)z = x(zy) \text{ and } (xz)y = x(yz) \\ &F'(x,y,z) : (xy)z = (xz)y \text{ and } x(yz) = x(zy). \end{aligned} \tag{1.2}$$

[We obtain the same three conditions by setting $y = 1$ or $z = 1$.]

[1]Some authors (ourselves included) often reserve the term Bol loop for a non-Moufang Bol loop, Moufang loop for a nonassociative Moufang loop, and so on. In this paper, however, we will be more permissive. Thus, for example, a group is a Bol loop; an associative ring is right alternative; etc.

Thus, if L is an SRAR loop, then, for all $x, y, z \in L$, at least one of the conditions $D'(x, y, z)$ or $E'(x, y, z)$ or $F'(x, y, z)$ holds. [Again, we will simply say that D' or E' or F' holds.]

Furthermore, as in Lemma 1.2, for any particular x, y, z, either all three of the conditions D', E', F' hold, or exactly one does.

We remark that while conditions D, E and F imply conditions D', E' and F', the converse is not true.

Example 1.3. *The loop shown in Table 1 is number* 16.7.2.1 *in Moorhouse's list of Bol loops of order* 16 [Moo]. *A computer search using Maple* [Maple] *or the LOOPS package* [LOOPS] *for GAP* [GAP] *shows that, for each $x, y, z \in L$, at least one of $D'(x, y, z)$ or $E'(x, y, z)$ or $F'(x, y, z)$ holds.*[2] *However, in this loop, $[(2\cdot 2)\cdot 3]\cdot 9 = 11$, $2\cdot[(2\cdot 3)\cdot 9] = 9$, $[(2\cdot 9)\cdot 3]\cdot 2 = 13$ and $2 \cdot [(9 \cdot 3) \cdot 2] = 16$, so that none of $D(2, 2, 3, 9)$, $E(2, 2, 3, 9)$ or $F(2, 2, 3, 9)$ holds.*

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1	4	3	8	7	6	5	10	11	12	9	15	13	16	14
3	4	1	2	6	5	8	7	11	12	9	10	16	15	14	13
4	3	2	1	7	8	5	6	12	9	10	11	14	16	13	15
5	8	6	7	1	3	4	2	16	14	13	15	11	10	12	9
6	7	5	8	3	1	2	4	13	15	16	14	9	12	10	11
7	6	8	5	4	2	1	3	15	16	14	13	10	9	11	12
8	5	7	6	2	4	3	1	14	13	15	16	12	11	9	10
9	10	13	12	16	11	15	14	5	7	6	8	3	4	2	1
10	9	15	11	14	12	13	16	8	5	7	6	4	1	3	2
11	12	16	10	13	9	14	15	6	8	5	7	1	2	4	3
12	11	14	9	15	10	16	13	7	6	8	5	2	3	1	4
13	15	9	14	11	16	10	12	3	2	1	4	5	8	7	6
14	16	12	13	10	15	11	9	2	1	4	3	7	5	6	8
15	13	10	16	12	14	9	11	4	3	2	1	8	6	5	7
16	14	11	15	9	13	12	10	1	4	3	2	6	7	8	5

Table 1.

Incidentally, this example answers Question 4.5 of [Goo99] in the negative.

In the current paper, we investigate connections between SRAR loops and RA2 loops. Amongst our observations are that for each pair of conditions D, E, F, any SRAR loop having the property that, for all x, y, z, w, at least one of the two conditions of the pair is satisfied is necessarily an RA2 loop. Actually, our main result is somewhat stronger. That is, for each pair of conditions D', E', F', any SRAR loop having the property that, for all x, y, z, at least one of the two conditions of the pair is satisfied is necessarily an RA2 loop.

[2]This computer search also shows that, of the 2033 non-Moufang Bol loops of order 16, there are 1873 that are SRAR. Of the 160 that are not, exactly 5 have the property that D', E' or F' hold.

Again, the converse is not true. That is, there are RA2 loops in which it is not true that there exists a pair of the conditions D, E, F (or even of the conditions D', E', F') such that every ordered set of four elements in the loop satisfies at least one condition of the pair. We will discuss this in more detail below (Example 2.3).

2. Terminology and Notation

If L is a loop, and if

$$[(xy)z]y = x[(yz)y] \qquad \text{right Bol identity} \tag{2.1}$$

holds for all $x, y, z \in L$, then L is called a *(right) Bol loop*. If

$$[(xy)z]y = x[y(zy)] \qquad \text{right Moufang identity} \tag{2.2}$$

holds for all $x, y, z \in L$, then L is called a *Moufang loop*. Taking $x = 1$ in equation (2.2), we see that Moufang loops satisfy the identity

$$(yz)y = y(zy) \qquad \text{flexible identity.}$$

Applying this to (2.2) shows that every Moufang loop is right Bol. The converse is not true: There exist Bol loops that are not Moufang.

Taking $z = 1$ in equation (2.1), we see that the *right alternative law*

$$(xy)y = x(yy) \qquad \text{right alternative law} \tag{2.3}$$

holds in any right Bol loop. Moreover, a right Bol loop satisfies the *right inverse property*:

$$(xy)y^{-1} = x \qquad \text{right inverse property.} \tag{2.4}$$

Although not obvious from the presentation above, a Moufang loop satisfies not only the right alternative law, but also

$$(xx)y = x(xy) \qquad \text{the left alternative law} \tag{2.5}$$

as well. In fact,

Theorem 2.1 (Moufang)**.** *If three (not necessarily distinct) elements of a Moufang loop associate (in any order), then the subloop which they generate is a group. (See* [Pfl90]*, Section IV.2.)*

Thus, in a Moufang loop, the left alternative law is a consequence of the right, Moufang loops satisfy the left inverse property, and so on. By contrast, a right Bol loop is left alternative or is a left inverse property loop if and only if it is Moufang. These important properties about Moufang loops all have ring theoretic analogues due primarily to Artin, who showed that if three (not necessarily distinct) elements of an alternative ring associate in any order, then the subring they generate is a group [GJM96]. Moreover, an alternative ring satisfies three (equivalent) Moufang identities (of which we have given just one), satisfied by any Moufang loop.

Much more can be said about Bol and Moufang loops, and about alternative rings. The interested reader is referred to [GJM96] and [Pfl90] for more information.

Given a commutative and associative ring R with unity and a loop L, one can construct the *loop ring* RL in a manner similar to that in which group rings are constructed. That is, RL is the free R-module with basis consisting of the elements of L and multiplication defined by extending the product in L by means of the distributive laws. Clearly if L is not associative, then neither is RL. Not necessarily associative rings that satisfy the right alternative law (2.3) are called right alternative rings and those that satisfy both (2.3) and (2.5) are called alternative rings. (Again, we refer the reader who would like to learn more about alternative rings to [GJM96].)

In [**?**], the first two authors proved the following theorem:

Theorem 2.2. *If R is a commutative (associative) ring with unity and $\operatorname{char} R = 2$, and if L is a Moufang loop, then the loop ring RL is alternative if and only if, for all $x, y, z \in L$,*

(i) *at least one of the following three conditions holds:*

$$\begin{aligned} &A(x,y,z) : (xy)z = x(yz) \ \textit{and} \ (yx)z = y(xz) \\ &B(x,y,z) : (xy)z = y(xz) \ \textit{and} \ x(yz) = (yx)z \\ &C(x,y,z) : (xy)z = (yx)z \ \textit{and} \ x(yz) = y(xz) \end{aligned} \tag{2.6}$$

and

(ii) *at least one of the following three conditions holds*[3]

$$\begin{aligned} &A^*(x,y,z) : (xy)z = x(yz) \ \textit{and} \ (xz)y = x(zy) \\ &B^*(x,y,z) : (xy)z = x(zy) \ \textit{and} \ x(yz) = (xz)y \\ &C^*(x,y,z) : (xy)z = (xz)y \ \textit{and} \ x(yz) = x(zy). \end{aligned} \tag{2.7}$$

The condition that at least one of the equations in (2.6) holds is equivalent to the left alternative law in RL and the condition that at least one of the equations in (2.7) holds is equivalent to the right alternative law. Loops that have alternative loop rings in characteristic 2 are known as *RA2 loops*[4].

Note that the conditions A^*, B^* and C^* are precisely the conditions which we have called D', E' and F' in the current paper. Thus, for a Moufang loop L, the loop ring RL is right alternative if and only if at least one of the conditions $D'(x,y,z)$, $E'(x,y,z)$ and $F'(x,y,z)$ holds for each x, y, z in L. In particular, this is true in any RA2 loop.

As stated in Section 1, in characteristic different from 2, right alternative loop rings are actually alternative [Kun98]. Since alternative rings satisfy the Moufang identity, in characteristic different from 2, any loop giving rise to a right alternative and hence alternative loop ring must be Moufang. In

[3]The notation here differs slightly from the notation in [**?**] so as not to cause confusion with the notation in [GR95] and with the notation in the present paper.

[4]Again, this usually refers to nonassociative loops, but we suppress that here.

characteristic 2, however, the underlying loop of a right alternative loop ring need not even be Bol. Together with D. A. Robinson, the second author studied those loops L for which RL satisfies just the right alternative law. They were particularly interested in loops for which RL is itself a Bol loop. They found that a loop L is of this type if and only if L is right Bol and, for all $x, y, z \in L$, at least one of the three conditions of equations (1.1) holds. (See Theorem 1.1.)

We stated above that there are RA2 loops in which it is not true that there exist a pair of the conditions D, E, F (or even of the conditions D', E', F') such that every ordered set of four (resp. three) elements in the loop satisfies at least one condition of the pair. The following is an example.

Example 2.3. *The smallest Moufang loop, ($M(S_3, 2)$ in the notation of* [C74] *or 12/1 in the notation of* [GMR99]*), has Cayley table shown in Table 2.*

1	2	3	4	5	6	7	8	9	10	11	12
2	3	1	5	6	4	8	9	7	12	10	11
3	1	2	6	4	5	9	7	8	11	12	10
4	6	5	1	3	2	10	11	12	7	8	9
5	4	6	2	1	3	11	12	10	9	7	8
6	5	4	3	2	1	12	10	11	8	9	7
7	9	8	10	11	12	1	3	2	4	5	6
8	7	9	11	12	10	2	1	3	6	4	5
9	8	7	12	10	11	3	2	1	5	6	4
10	11	12	7	9	8	4	6	5	1	2	3
11	12	10	8	7	9	5	4	6	3	1	2
12	10	11	9	8	7	6	5	4	2	3	1

TABLE 2.

*This is an RA2 loop ([**?**], Theorem 5.4).*

Consider the three triples of elements $\{x = 2, y = 3, z = 8\}$, $\{x = 2, y = 5, z = 9\}$ *and* $\{x = 2, y = 4, z = 10\}$.

$$\begin{array}{llll}
(2\cdot 3)\cdot 8 = 8 & 2\cdot(3\cdot 8) = 8 & (2\cdot 8)\cdot 3 = 7 & 2\cdot(8\cdot 3) = 7\\
(2\cdot 5)\cdot 9 = 11 & 2\cdot(5\cdot 9) = 12 & (2\cdot 9)\cdot 5 = 11 & 2\cdot(9\cdot 5) = 12\\
(2\cdot 4)\cdot 10 = 9 & 2\cdot(4\cdot 10) = 8 & (2\cdot 10)\cdot 4 = 9 & 2\cdot(10\cdot 4) = 8
\end{array}$$

so that

$D'(2,3,8)$ *holds but* $E'(2,3,8)$ *and* $F'(2,3,8)$ *do not;*
$E'(2,5,9)$ *holds but* $D'(2,5,9)$ *and* $F'(2,5,9)$ *do not;*
$F'(2,4,10)$ *holds but* $D'(2,4,10)$ *and* $E'(2,4,10)$ *do not.*

Thus, there is no pair of the conditions D', E' *and* F' *such that one of the pair holds for each triple of elements in the loop.*

[Note that the same argument works for D, E *and* F *by using the 4-tuples* $\{2,3,8,1\}$, $\{2,5,9,1\}$ *and* $\{2,4,10,1\}$ *respectively.]*

3. Main Results

Throughout the remainder of this paper, R will denote a commutative (and associative) ring with unity and of characteristic 2, and L will denote a (right) Bol loop. Also, we will label the six equations of (1.2) as D_1', D_2', E_1', E_2', F_1' and F_2' respectively.

Proposition 3.1. *If L is an RA2 loop, then L is SRAR.*

Proof. If L is RA2, then RL is alternative, so RL satisfies the Moufang and hence right Bol identity. □

As noted earlier, the converse in not true, as an RA2 loop is necessarily Moufang but an SRAR loop need only be Bol. The main goal of this paper is to exhibit conditions under which an SRAR loop is indeed RA2.

We begin with the following lemma.

Lemma 3.2. *If L is a Bol loop then, for $x, y \in L$, $x^{-1}(xy) = y$ if and only if $x(x^{-1}y) = y$.*

Proof. Assume $x^{-1}(xy) = y$. By the right inverse property, $x^{-1} = y(xy)^{-1}$, and so, using (2.1), we have $x(x^{-1}y) = x\{[y(xy)^{-1}]y\} = [(xy)(xy)^{-1}]y = y$. The converse follows from switching the roles of x and x^{-1}. □

Lemma 3.3. *Let L be a Bol loop with the property that, for all $x, y \in L$, $xy = yx$ or $x^{-1}(xy) = y$. Then L is a Moufang loop.*

Proof. We will show that L satisfies the left inverse property. That is, we will show that $x^{-1}(xy) = y$ holds for all $x, y \in L$.

Let $x, y \in L$. If $x^{-1}(xy) = y$, we are done, so, by the hypothesis of the theorem, there is no loss of generality if we assume that $xy = yx$.

By the right Bol identity (2.1),

$$x\{[y(xy)^{-1}]y\} = [(xy)(xy)^{-1}]y = y, \tag{3.1}$$

and by (2.1) and the right inverse property, (2.4),

$$x\{[x^{-1}(yx)]x^{-1}\} = [(xx^{-1})(yx)]x^{-1} = (yx)x^{-1} = y, \tag{3.2}$$

so

$$x\{[y(xy)^{-1}]y\} = x\{[x^{-1}(yx)]x^{-1}\}. \tag{3.3}$$

Canceling the x on the left and then multiplying by x on the right, we get

$$\{[y(xy)^{-1}]y\}x = \{[x^{-1}(yx)]x^{-1}\}x. \tag{3.4}$$

By the right inverse property, (2.4), the right hand side of (3.4) is equal to $x^{-1}(yx) = x^{-1}(xy)$.

Thus $\{[y(xy)^{-1}]y\}x = x^{-1}(xy)$. If x and $\{[y(xy)^{-1}]y\}$ commute, this becomes $x\{[y(xy)^{-1}]y\} = x^{-1}(xy)$. By (3.1), the left hand side of this equation is equal to y, so that $y = x^{-1}(xy)$. On the other hand, if x and $\{[y(xy)^{-1}]y\}$ do not commute, then by the hypothesis of the Lemma,

$$x^{-1}(x\{[y(xy)^{-1}]y\}) = [y(xy)^{-1}]y. \tag{3.5}$$

But, by (3.1), $x\{[y(xy)^{-1}]y\} = y$, so that (3.5) becomes $x^{-1}y = [y(xy)^{-1}]y$. Canceling the y's on the right and using the right inverse property, (2.4), this becomes $x^{-1}(xy) = y$.

Thus, we have shown that for every $x, y \in L$, $x^{-1}(xy) = y$. But this is just the left inverse property, and a right Bol loop that satisfies the left inverse property is Moufang. So L is Moufang. □

Before stating the main theorem of this paper, we alert the reader to the fact that loops that satisfy the identity $[(xy)z]x = x[y(zx)]$ are known as *extra loops* [Fen68]. These loops have been characterized by the first author and D. A. Robinson as those Moufang loops in which all squares lie in the nucleus [CR72].

Theorem 3.4. *Let L be a Bol loop. If, for each $x, y, z \in L$,*

(1) *$D'(x, y, z)$ or $E'(x, y, z)$ holds, then L is an RA2 loop that is extra,*
(2) *$D'(x, y, z)$ or $F'(x, y, z)$ holds, then L is a group,*
(3) *$E'(x, y, z)$ or $F'(x, y, z)$ holds, then L is an abelian group.*

Proof. (1) First, we verify that L is a Moufang loop. If, for each $x, y, z \in L$, $D'(x, y, z)$ or $E'(x, y, z)$ holds, then, in particular, for each $x, y \in L$, $D'_1(x^{-1}, xy, x)$ or $E'_2(x^{-1}, xy, x)$ holds. Thus

$$[x^{-1}(xy)]x = x^{-1}[(xy)x] \qquad \text{or} \qquad xy = (x^{-1}x)xy = x^{-1}[(xy)x] \tag{3.6}$$

for all $x, y \in L$. By (2.1), $x^{-1}[(xy)x] = [(x^{-1}x)y]x = yx$. Using this in both parts of (3.6), and then canceling x's in the left equation, we obtain

$$x^{-1}(xy) = y \qquad \text{or} \qquad xy = yx$$

for all $x, y \in L$. Therefore, by Lemma 3.3, L is Moufang.

Since $D'(x, y, z)$ or $E'(x, y, z)$ holds for all $x, y, z \in L$, it is certainly true that $D'(x, y, z)$ or $E'(x, y, z)$ or $F'(x, y, z)$ holds for all $x, y, z \in L$. Therefore by the remark following Theorem 2.2, RL is right alternative for any ring R of characteristic 2. But since L is Moufang, then RL is also left alternative [CG88], and hence RA2.

To complete the proof that L is an extra loop, we will show that the nucleus of L contains every square. For each $x, y, z \in L$, either $D'_1(x, y, z)$ or $E'_1(x, y, z)$ holds, so

$$(xy)z = x(yz) \qquad \text{or} \qquad (xy)z = x(zy). \tag{3.7}$$

Note that saying that $D'_1(x, y, z)$ holds is equivalent to saying that $(x, y, z) = 1$. Therefore, by Moufang's Theorem (Theorem 2.1), if u, v, w generate the same subloop of L as do x, y, z, then $D'_1(u, v, w)$ holds if and only if $D'_1(x, y, z)$ holds. Thus, in particular, if $D'_1(x, y, z)$ fails to hold, then the same is true for $D'_1(y^{-1}, yz, x)$ and $D'_1(y, yz, x)$. In this case, $E'_1(y^{-1}, yz, x)$ and $E'_1(y, yz, x)$ must both hold.

Let x, y, z be three arbitrary elements of L.

If $D'_1(x, y, z)$ holds, then x, y and z generate a group, and so, in particular, $(y^2, z, x) = 1$.

On the other hand, if $D_1'(x,y,z)$ fails to hold, then, as we saw above, $E_1'(y^{-1},yz,x)$ must hold. That is,

$$[y^{-1}(yz)]x = y^{-1}[x(yz)]. \tag{3.8}$$

By the left inverse property, (3.8) becomes $zx = y^{-1}[x(yz)]$. Now multiply on the left by y and then use the left inverse property again to get

$$y(zx) = x(yz). \tag{3.9}$$

Similarly, $E_1'(y,yz,x)$ must also hold, so that

$$[y(yz)]x = y[x(yz)]. \tag{3.10}$$

Using (3.9) to substitute for $x(yz)$ and left alternativity (L is Moufang) twice, we get

$$(y^2z)x = [y(yz)]x = y[x(yz)] = y[y(zx)] = y^2(zx). \tag{3.11}$$

Thus, for any $x,y,z \in L$, regardless of whether they associate or not, $(y^2,z,x) = 1$, and so squares are in the left nucleus. Since the nuclei of a Moufang loop coincide, squares are in the nucleus and the loop is extra. This completes the proof of (1).

(2) As in case (1), we first verify that L is a Moufang loop. If, for each $x,y,z \in L$, $D'(x,y,z)$ or $F'(x,y,z)$ holds, then for each $x,y \in L$, $D_1'(x,x^{-1},y)$ or $F_1'(x,x^{-1},y)$ holds. Thus,

$$y = x(x^{-1}y) \qquad \text{or} \qquad y = (xy)x^{-1}.$$

Applying Lemma 3.2 to the first equation here and the right inverse property to the second, we obtain

$$y = x^{-1}(xy) \qquad \text{or} \qquad xy = yx.$$

This holds for all $x,y \in L$. Therefore, L is Moufang by Lemma 3.3.

Now, we show that L is associative. For any $x,y,z \in L$, if $D_1'(x,y,z)$ holds, there is nothing to show, so assume $F'(x,y,z)$ holds. Again applying Moufang's Theorem, we also have $F'(z,x,y)$, $F'(y,x,z)$ and $F'(z,yz,x)$.

From $F_2'(z,x,y)$, we get

$$xy = yx; \tag{3.12}$$

from $F_1'(y,x,z)$, we get

$$(yx)z = (yz)x; \tag{3.13}$$

and, from $F_2'(z,yz,x)$, we get

$$(yz)x = x(yz). \tag{3.14}$$

Putting these together, we get

$$(xy)z = (yx)z = (yz)x = x(yz). \tag{3.15}$$

Thus, for any $x,y,z \in L$, $(x,y,z) = 1$, and so L is associative.

This completes part (2).

(3) Taking $x = 1$ in $E'(x, y, z)$ and $F'(x, y, z)$, we obtain that L is commutative and hence Moufang. Take $x, y, z \in L$. Either $(xy)z = x(zy) = x(yz)$ by $E(x, y, z)$ and commutativity, or $z(xy) = (xy)z = (xz)y = (zx)y$ by commutativity and $F(x, y, z)$. In either case, x, y and z associate. Thus L is an abelian group. □

We now can answer our question.

Corollary 3.5. *If L is an SRAR loop in which one of the following is true:*

(1) *$D'(x, y, z)$ or $E'(x, y, z)$ holds for all $x, y, z \in L$, or*
(2) *$D'(x, y, z)$ or $F'(x, y, z)$ holds for all $x, y, z \in L$, or*
(3) *$E'(x, y, z)$ or $F'(x, y, z)$ holds for all $x, y, z \in L$,*

then L is a (possibly associative) RA2 loop.

Proof. Since groups have associative and hence alternative loop rings, the corollary is an immediate consequence of Theorem 3.4 □

Note that Example 2.3 provides a counterexample to the converse of this corollary. This leaves open the question of finding a necessary and sufficient condition for an SRAR loop to be RA2.

Theorem 3.4 also provides information about the possible order of a nonassociative SRAR loop.

Corollary 3.6. *An SRAR loop of odd order must be associative.*

Proof. We first show that and SRAR loop of odd order must be Moufang. By Lemma 1.2, for any $x, y, z \in L$, either $D'(x, y, z)$ holds, or $E'(x, y, z)$ holds, or $F'(x, y, z)$ holds.

Suppose that x, y, z is a triple that satisfies $F'(x, y, z)$. That is, $(xy)z = (xz)y$ and $yz = zy$. Then, since an SRAR loop satisfies the right Bol identity,

$$(xz)y^2 = [(xz)y]y = [(xy)z]y = x[(yz)y] = x[(zy)y] = x(zy^2),$$

and so x, z and y^2 associate in that order. But since L is of odd order, so is y. Thus $y = (y^2)^r$ for some r, and so x, z and y associate in that order. That is, $(xz)y = x(zy)$.

But then, we also have $(xy)z = (xz)y = x(zy) = x(yz)$, so that $D'(x, y, z)$ holds.

This says that, for any triple x, y, z in L, either $D'(x, y, z$ must hold or $E'(x, y, z)$ holds. But then, by Theorem 3.4, L is an extra loop and hence is Moufang.

But, by Theorem 1.3 of [CG88], if L is Moufang and RL is right alternative, then RL is an alternative ring. Hence, by Corollary 2.5 of [CG90], L is associative.

□

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